



Meshless Methods

Soheil Mohammadi

Professor of Computational Mechanics

School of Civil Engineering

University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: +98-912-5030 392

Email: smoham@ut.ac.ir, <http://eng.ut.ac.ir/civ/smohammadi>

SPH Method (1968)

- Astrophysical Simulations
 - Lucy (1977,1982)
 - Gingold, Monaghan (1977,1982,1985)
- Some present astrophysical problems:
 - Modelling of universe (1990)
 - Collapse/formation of galaxies (1991-2000)
 - Collapse of black holes with neutron stars (1998-2000)
 - Simulations of supernovas (1999)
 - Single/multiple detonations in white dwarfs (1999)

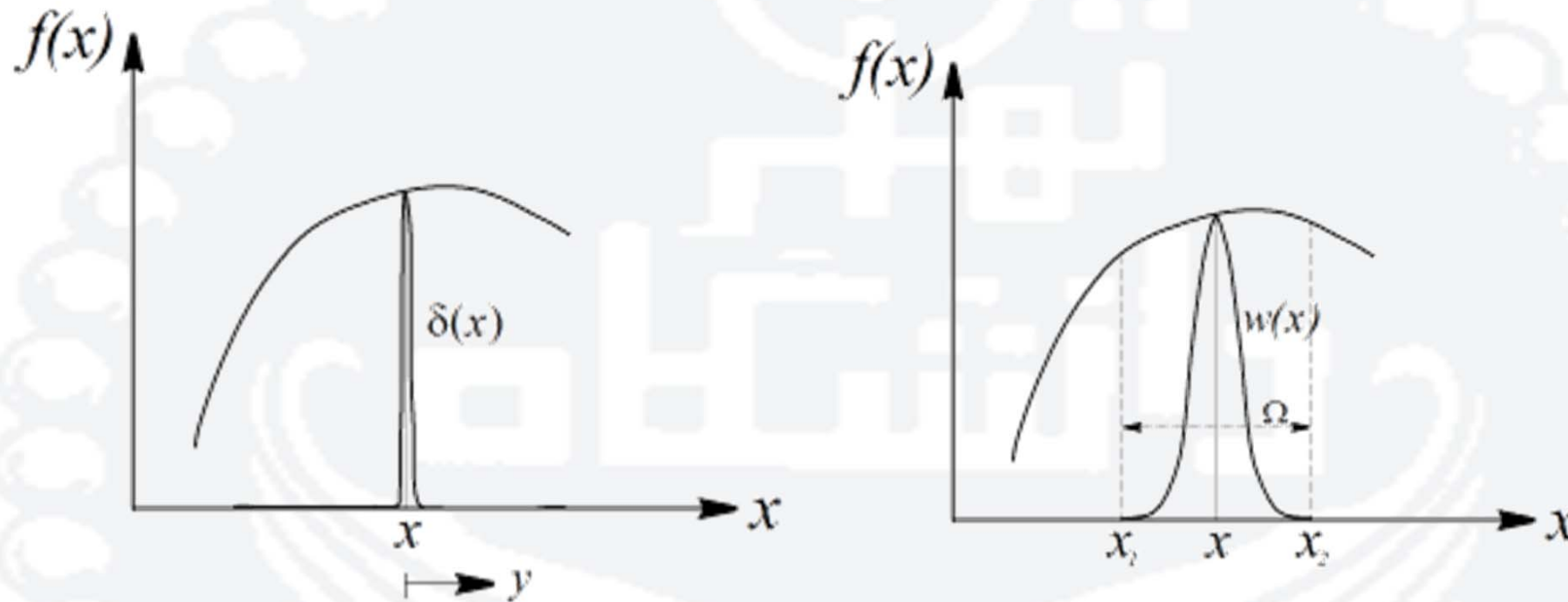
SPH Method

- Astrophysical Simulations
 - In astrophysical applications, the real physical system is discrete, and in order to avoid singularity, a local continuous field is generated by introducing a localized kernel function.
 - The kernel function can be interpreted as:
 - A smoothing interpolation field
 - The probability of a particle's position

SPH Method

- Mechanical Simulations
 - In mechanics, a continuum system is discretized into a discrete algebraic system.
 - Hydrodynamics is interpreted as Mechanics
 - The method is also called “Smoothed Particle Applied Mechanics”

Interpolation by Kernel Estimates:

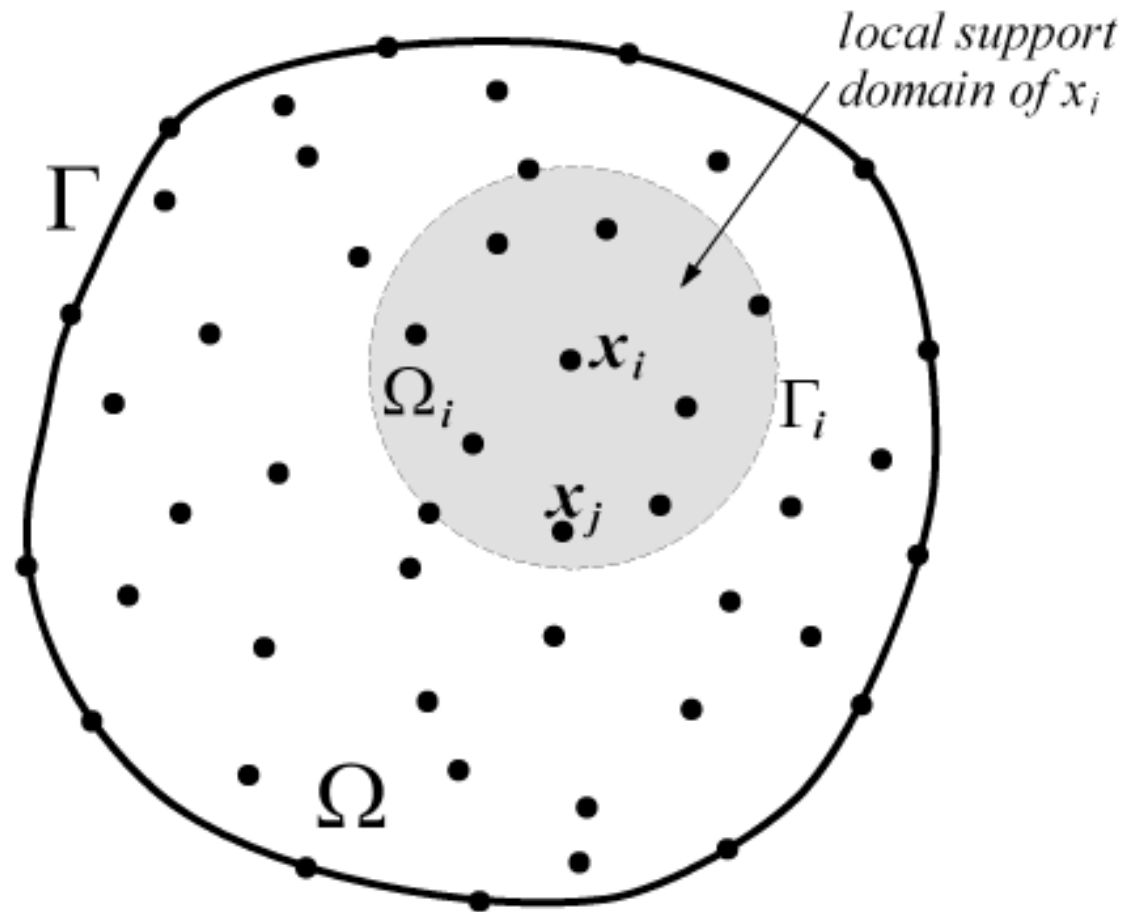


$$u_h(x) = \int_{\Omega} W(x-y, h) u(y) dy$$

$W(x-y, h) = W_h(x-y)$ = kernel/smoothing/weight function

h = the smoothing length, controlling the size of the compact support domain (smoothing or influence domain)

Smoothed Particle Hydrodynamics (FSPH):



Smoothed Particle Hydrodynamics (SPH)

- In the SPH method , the kernel approximation

$$u^h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

is approximated by imagining that the matter m is divided into N small volume elements with masses $m_1, m_2, \dots, m_n$ (nodal quadrature/particle approximation).

$$u^h(x) = \sum_{I \in Q(x)} W_h(x-x_I)u(x_I)\Delta V_I$$

$$u^h(x) = \sum_{I \in Q(x)} \phi_I(x)u(x_I)$$

$$\phi_I(x) = W_h(x-x_I)\Delta V_I \quad (\text{SPH shape function})$$

Smoothed Particle Hydrodynamics (SPH)

- The SPH approximation

$$u^h(x_i) = \sum_{j=1}^n u_j W_h(x_i - x_j) \Delta V_j$$

$$u^h(x_i) = \sum_{j=1}^n u_j W_{ij} \Delta V_j$$

- and its derivative

$$\frac{\partial u}{\partial x} = - \sum_{j=1}^n u_j \frac{\partial W_{ij}}{\partial x} \Delta V_j$$

- which is proved to be equivalent to:

$$\frac{\partial u}{\partial x} = \sum_{j=1}^n (u_i - u_j) \frac{\partial W_{ij}}{\partial x} \Delta V_j$$

Smoothed Particle Hydrodynamics (SPH)

- CSPM approximations:

$$u^h(x_i) = \frac{\sum_{j=1}^n u_j W_{ij} \Delta V_j}{\sum_{j=1}^n W_{ij} \Delta V_j}$$

$$\left(\frac{\partial u}{\partial x} \right)_i = \frac{\sum_{j=1}^n (u_j - u_i) W_{ij} \Delta V_j}{\sum_{j=1}^n (x_j - x_i) W_{ij} \Delta V_j}$$

Smoothed Particle Hydrodynamics (SPH)

- All spatial derivatives can be transformed into algebraic terms using the SPH approximation. As a result, the PDE is transformed into a set of nonlinear algebraic simultaneous equations.
- Relaxation methods are usually preferred in solving the set of nonlinear simultaneous equations.
- Only the original SPH and sometimes CSPM may easily transform the PDE to algebraic equations. Other sophisticated though complicated corrected SPH methods lack such a major advantage.
- Corrected SPH methods are usually adopted within a weak form formulation.

1D Elastic Wave (SPH)

- The governing equations for a 1D elastic bar subjected to elastic waves:

$$\dot{v} = \frac{1}{\rho} \sigma_{,x}$$

$$\dot{\sigma} = E v_{,x}$$

- SPH approximation:

$$\dot{v}_i = \frac{1}{\rho} (-1) \sum_{j=1}^n \sigma_j \frac{\partial W_{ij}}{\partial x} \Delta x_j$$

$$\dot{\sigma}_i = E (-1) \sum_{j=1}^n v_j \frac{\partial W_{ij}}{\partial x} \Delta x_j$$

1D Elastic Wave (SPH)

- Modified SPH approximation:

$$\dot{v}_i = \frac{1}{\rho} \sum_{j=1}^n (\sigma_i - \sigma_j) \frac{\partial W_{ij}}{\partial x} \Delta x_j$$

$$\dot{\sigma}_i = E \sum_{j=1}^n (v_i - v_j) \frac{\partial W_{ij}}{\partial x} \Delta x_j$$

- CSPM approximation

$$\dot{v}_i = \frac{1}{\rho} \frac{\sum_{j=1}^n (\sigma_j - \sigma_i) W_{ij} \Delta x_j}{\sum_{j=1}^n (x_j - x_i) W_{ij} \Delta x_j}$$

$$\dot{\sigma}_i = E \frac{\sum_{j=1}^n (v_j - v_i) W_{ij} \Delta x_j}{\sum_{j=1}^n (x_j - x_i) W_{ij} \Delta x_j}$$

1D Elastic Wave (SPH)

- Stabilized SPH approximation:

$$\dot{\sigma} = E v_{,x}$$

$$\dot{v} = \frac{1}{\rho} (\sigma - \Pi)_{,x} \quad \Pi = -0.2 \rho c h v_{,x} + 4 \rho h^2 (v_{,x})^2, v_{,x} < 0$$

then the SPH approximation should also include the $v_{,x}$ within the stabilized term.

- The solution proceeds according to the time integration scheme:

$$v_i^{n+1} = v_i^n + \dot{v}_i^n \Delta t$$

$$\sigma_i^{n+1} = \sigma_i^n + \dot{\sigma}_i^n \Delta t$$

2D Solid Mechanics (SPH)

2D governing equations for solid mechanics problems are:

- Equilibrium equations (momentum)

$$\sigma_{xx,x} + \sigma_{xy,y} = \rho \dot{v}_x$$

$$\sigma_{yx,x} + \sigma_{yy,y} = \rho \dot{v}_y$$

- Compatibility equations:

$$\dot{\epsilon}_{xx} = \dot{u}_{x,x} = v_{x,x}$$

$$\dot{\epsilon}_{yy} = \dot{u}_{y,y} = v_{y,y}$$

$$\dot{\epsilon}_{xy} = \frac{1}{2} (\dot{u}_{x,y} + \dot{u}_{y,x}) = \frac{1}{2} (v_{x,y} + v_{y,x})$$

- Material model:

$$\dot{\sigma} = D \dot{\epsilon}$$

2D Solid Mechanics (SPH)

The first two sets can be approximated by SH:

- Equilibrium equations

$$\rho \dot{v}_x = \sum_{j=1}^n (\sigma_{xx}^i - \sigma_{xx}^j) \frac{\partial W_{ij}}{\partial x} \Delta V_j + \sum_{j=1}^n (\sigma_{xy}^i - \sigma_{xy}^j) \frac{\partial W_{ij}}{\partial y} \Delta V_j$$
$$\rho \dot{v}_y = \sum_{j=1}^n (\sigma_{yx}^i - \sigma_{yx}^j) \frac{\partial W_{ij}}{\partial x} \Delta V_j + \sum_{j=1}^n (\sigma_{yy}^i - \sigma_{yy}^j) \frac{\partial W_{ij}}{\partial y} \Delta V_j$$

- Compatibility equations:

$$\dot{\epsilon}_x = \sum_{j=1}^n (v_x^i - v_x^j) \frac{\partial W_{ij}}{\partial x} \Delta V_j$$
$$\dot{\epsilon}_y = \sum_{j=1}^n (v_y^i - v_y^j) \frac{\partial W_{ij}}{\partial y} \Delta V_j$$
$$\dot{\epsilon}_{xy} = \frac{1}{2} \left[\sum_{j=1}^n (v_x^i - v_x^j) \frac{\partial W_{ij}}{\partial y} \Delta V_j + \sum_{j=1}^n (v_y^i - v_y^j) \frac{\partial W_{ij}}{\partial x} \Delta V_j \right]$$

2D Solid Mechanics (SPH)

The solution procedure is:

- Determine the acceleration from equilibrium equations.
- Compute the velocity by integrating acceleration.

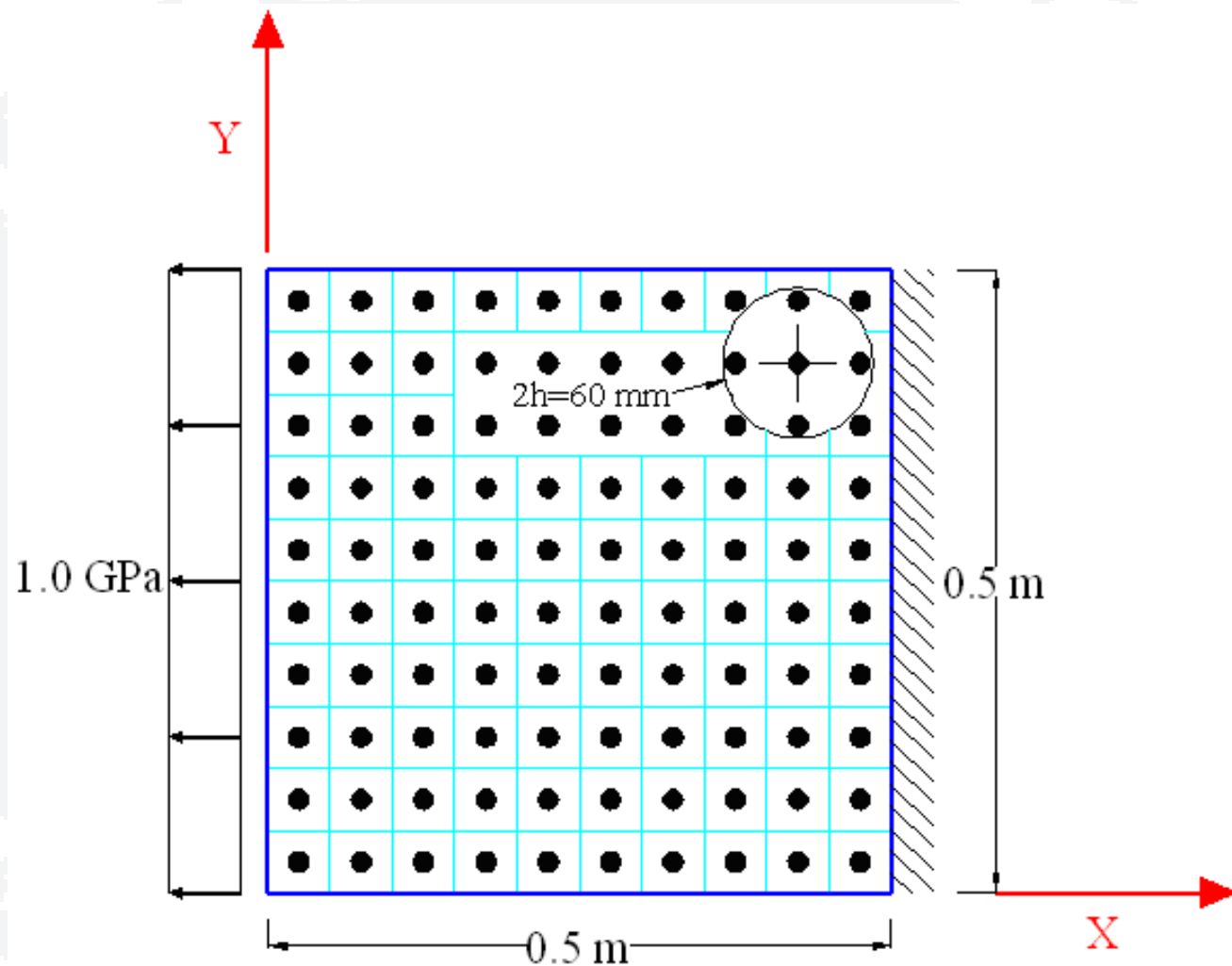
$$\mathbf{v}^{n+1} = \mathbf{v}^n + \dot{\mathbf{v}}^n \Delta t$$

- Determine strain rates from the compatibility equations.
- Compute the strain components

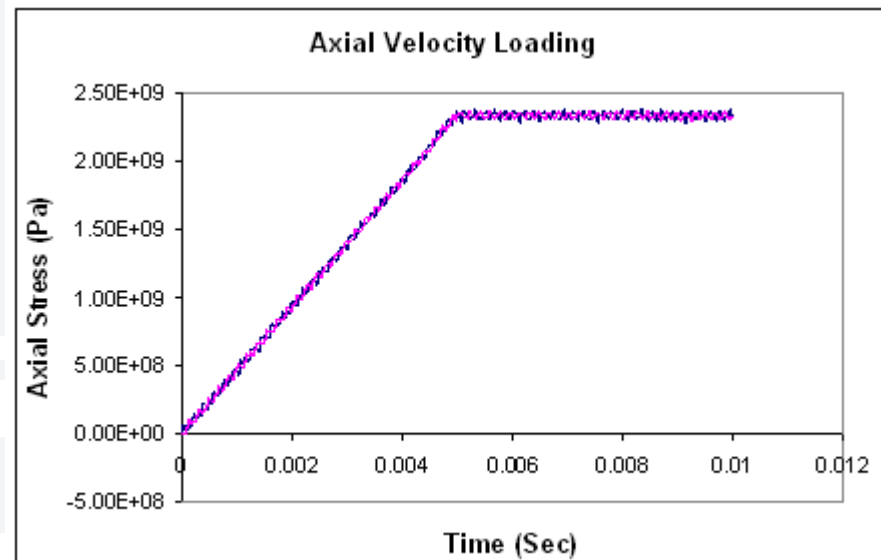
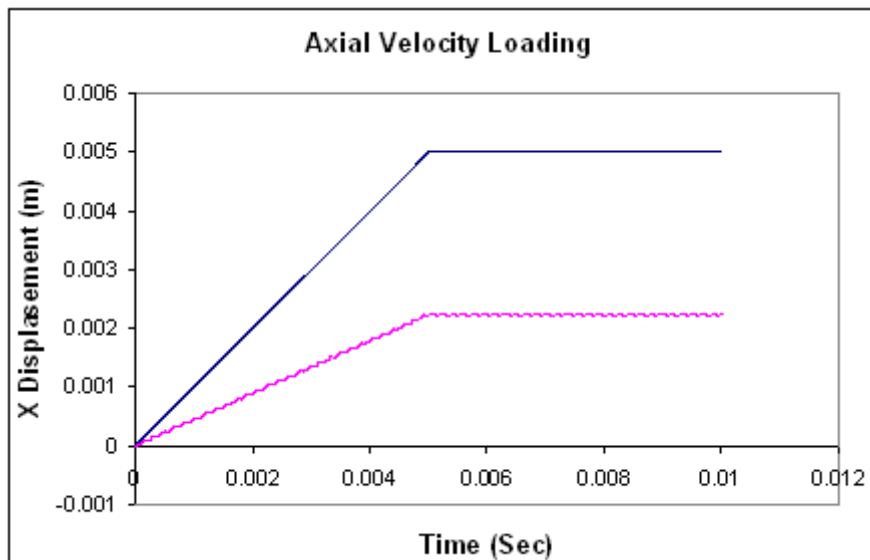
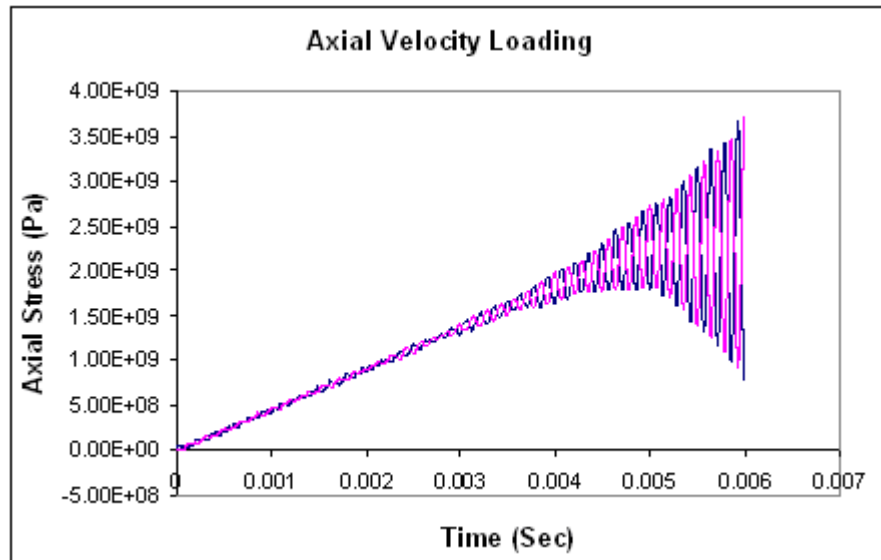
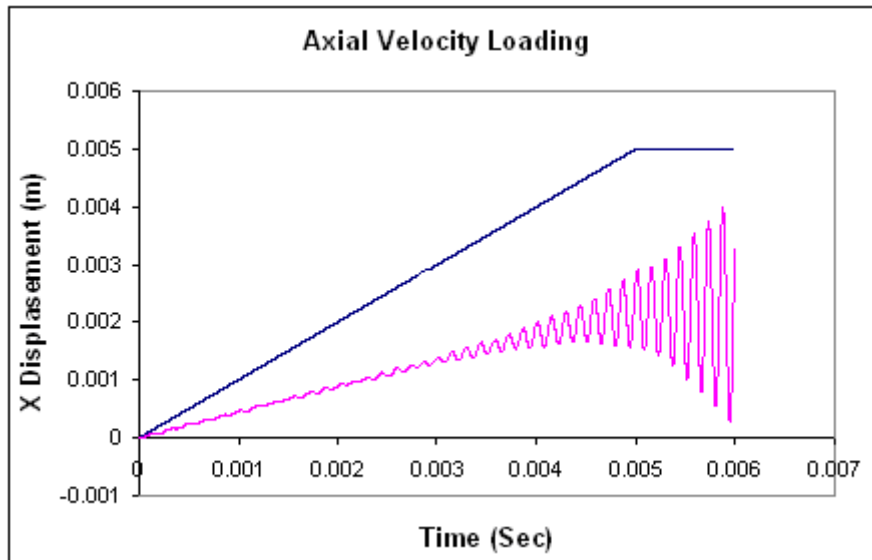
$$\boldsymbol{\varepsilon}^{n+1} = \boldsymbol{\varepsilon}^n + \dot{\boldsymbol{\varepsilon}}^n \Delta t$$

- Compute the stress from material model
- Go to the next time step with updated values.

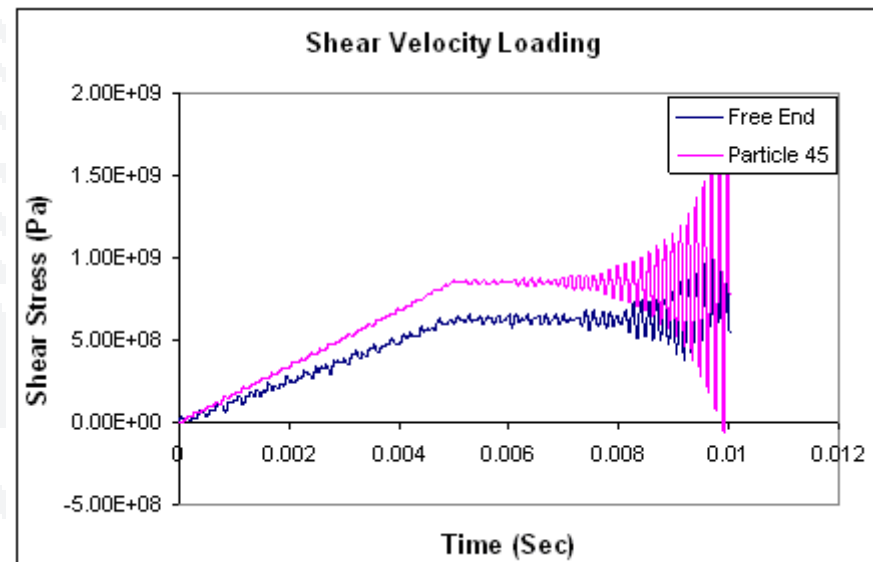
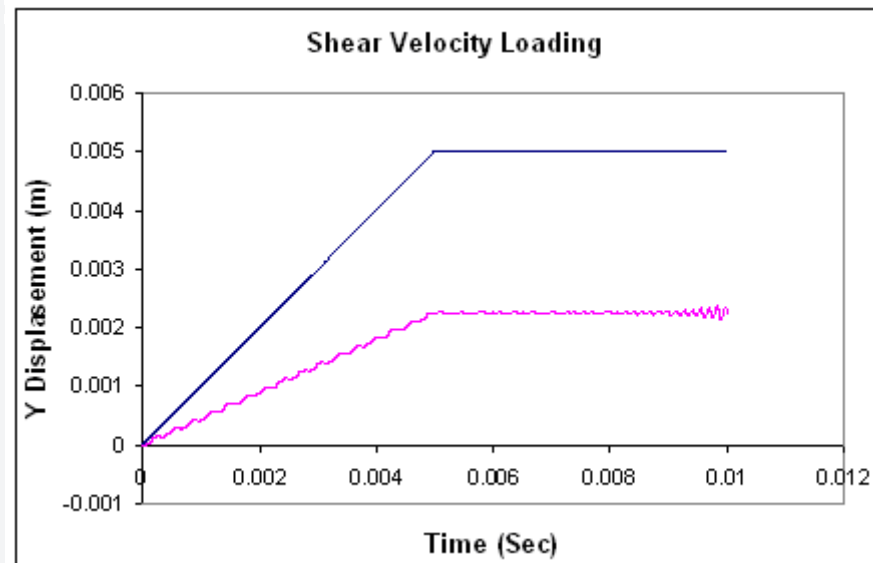
2D Solid Mechanics (SPH)



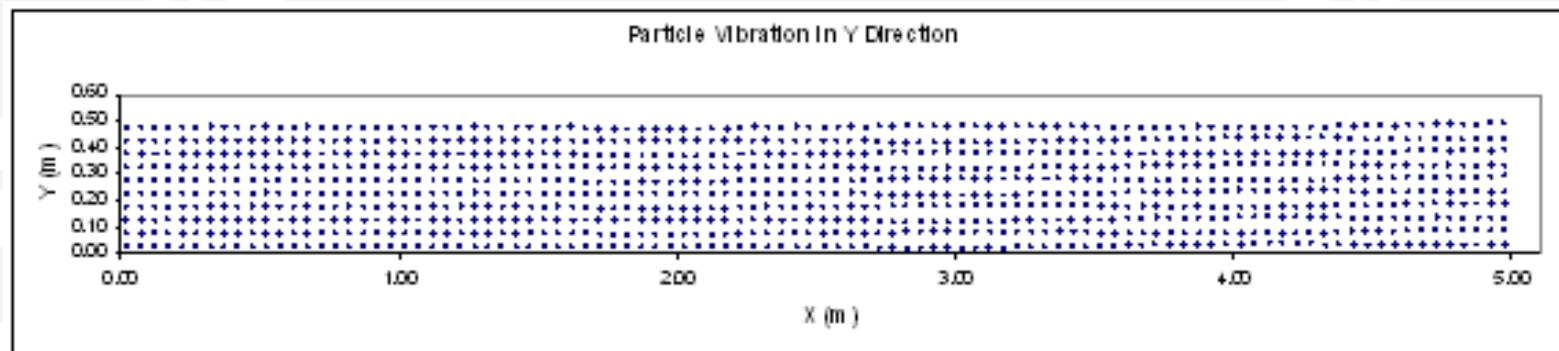
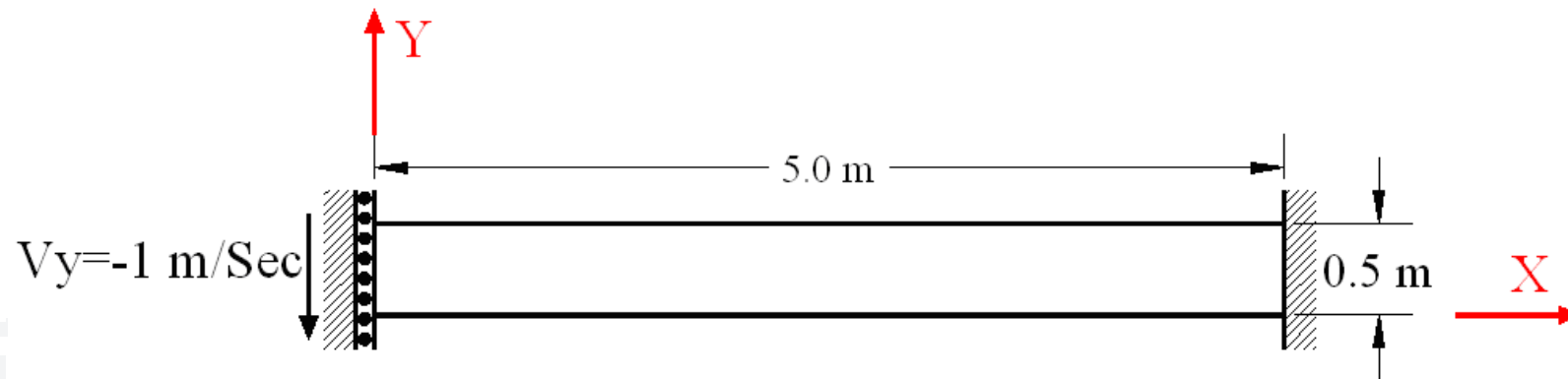
2D Solid Mechanics (SPH)



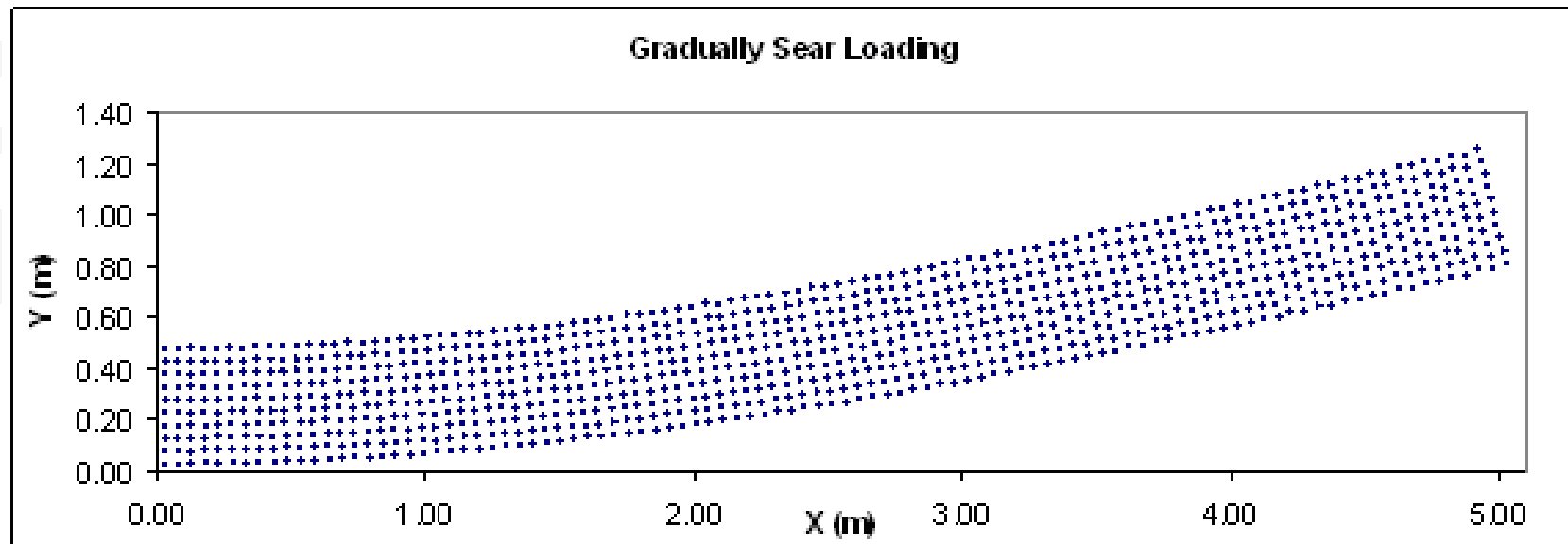
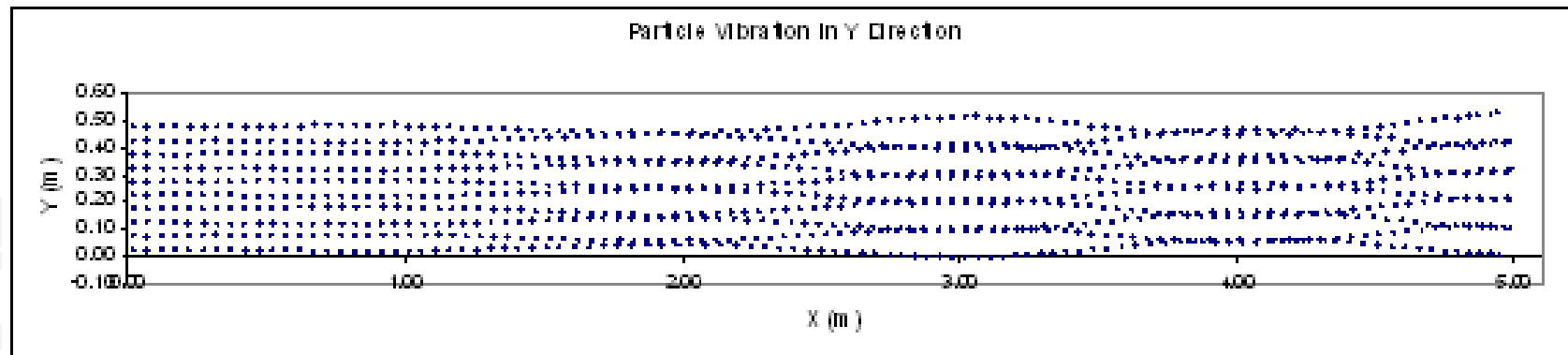
2D Solid Mechanics (SPH)



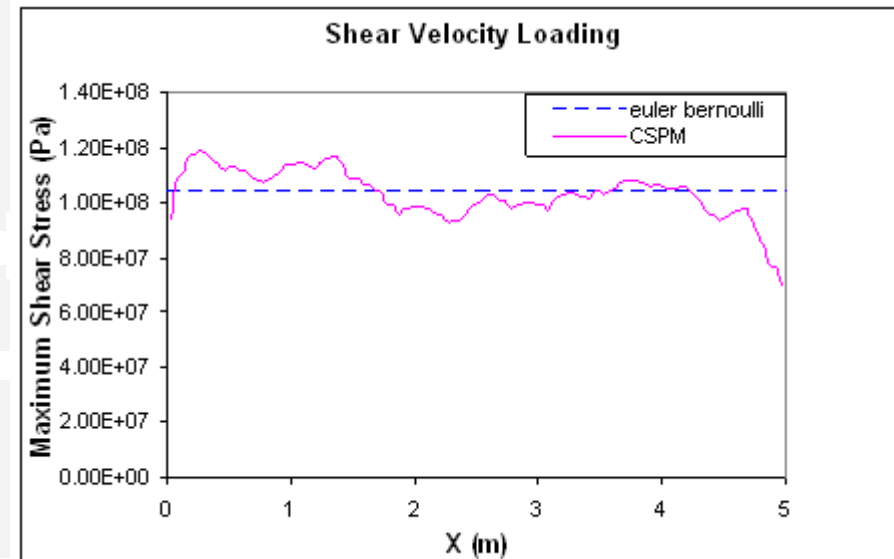
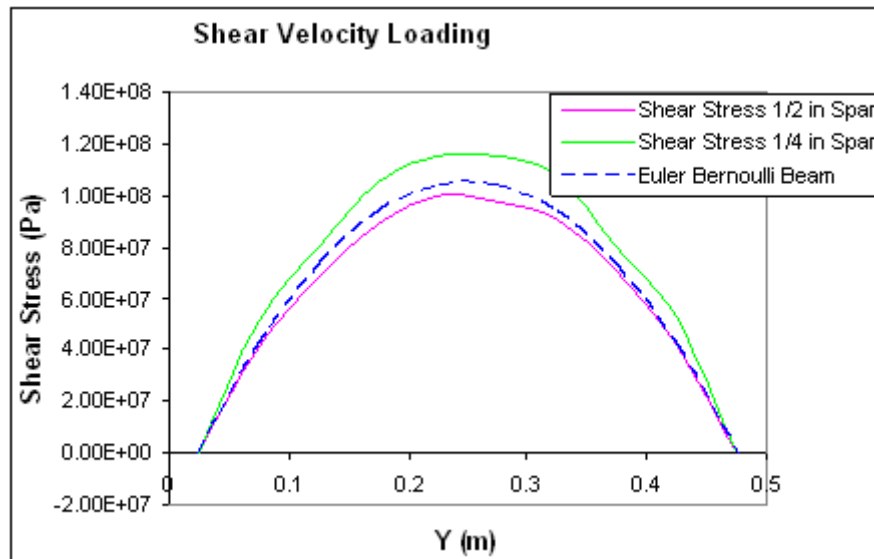
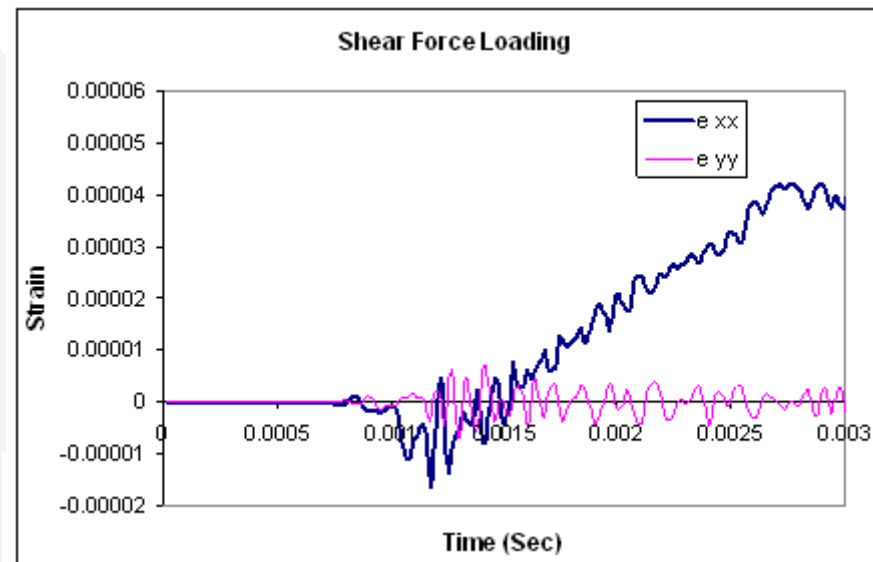
2D Solid Mechanics (SPH)



2D Solid Mechanics (SPH)



2D Solid Mechanics (SPH)



Simulation of Explosion (SPH)

- Conservation of mass

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}$$

- Conservation of momentum

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p$$

- Conservation of energy

$$\frac{Du}{Dt} = -\frac{p}{\rho} \nabla \cdot \mathbf{v}$$

- Equation of state

$$p = p(\rho, u)$$

- u = internal energy

- p = pressure

- $\mathbf{v}$ = velocity

$$p = A\left(1 - \frac{\omega\theta}{R_1}\right)e^{\frac{-R_1}{\theta}} + B\left(1 - \frac{\omega\theta}{R_2}\right)e^{\frac{-R_2}{\theta}} + \omega\theta\rho_0 E$$

$$p = (\gamma - 1)\rho E$$

Simulation of Explosion (SPH)

- Conservation of mass

$$\frac{D\rho_i}{Dt} = \sum_{j=1}^n m_j (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla_i W_{ij} \quad \text{or} \quad \frac{D\rho_i}{Dt} = \sum_{j=1}^n m_j W_{ij}$$

- Conservation of momentum

$$\frac{D\mathbf{v}_i}{Dt} = - \sum_{j=1}^n m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) \nabla_i W_{ij}$$

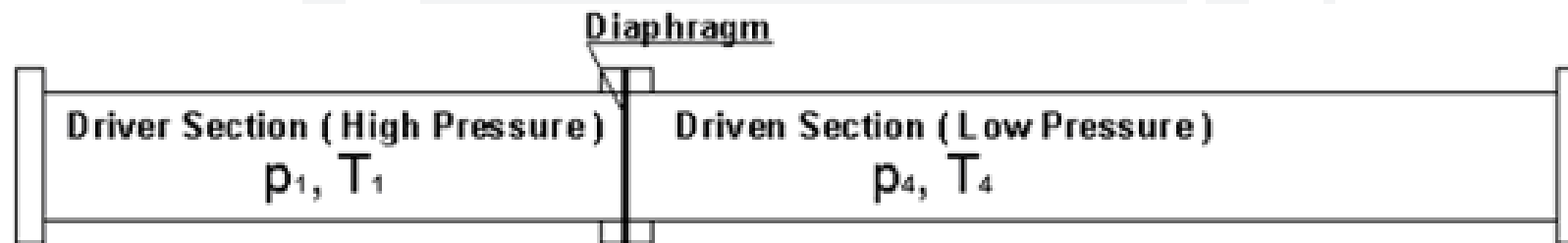
- Conservation of energy

$$\frac{Du_i}{Dt} = \frac{1}{2} \sum_{j=1}^n m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla_i W_{ij}$$

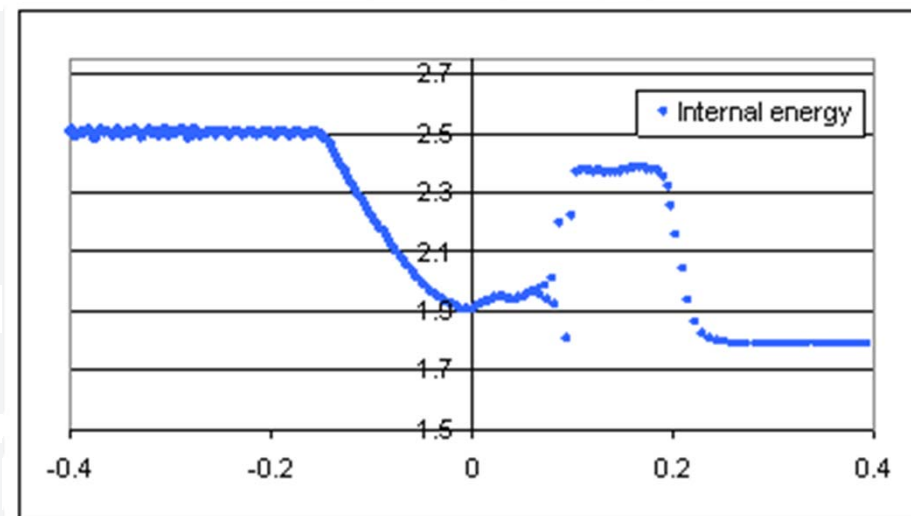
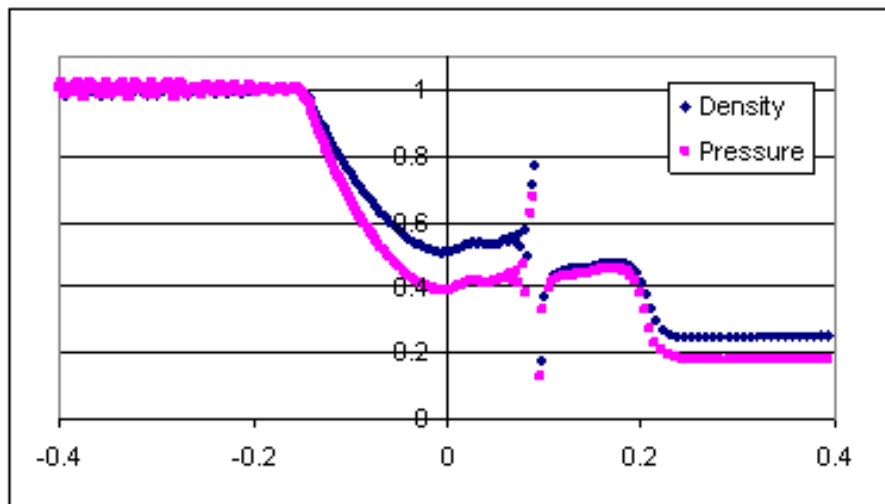
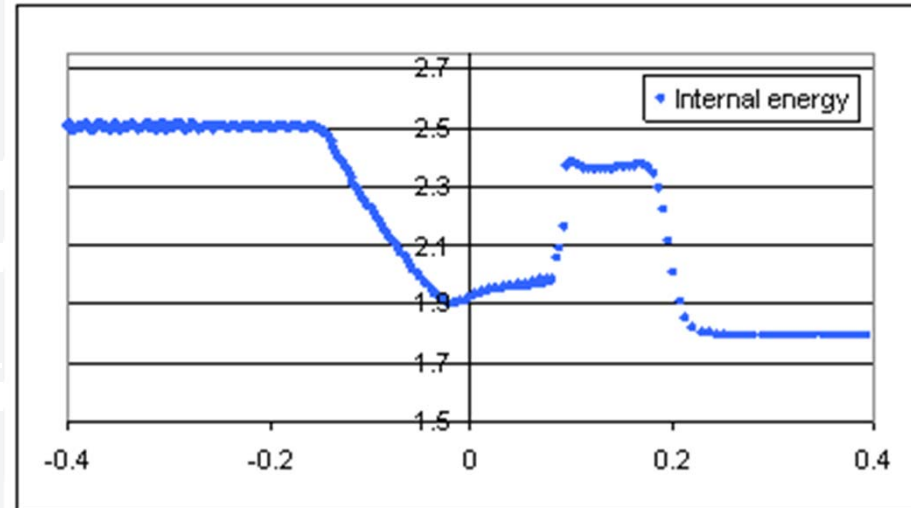
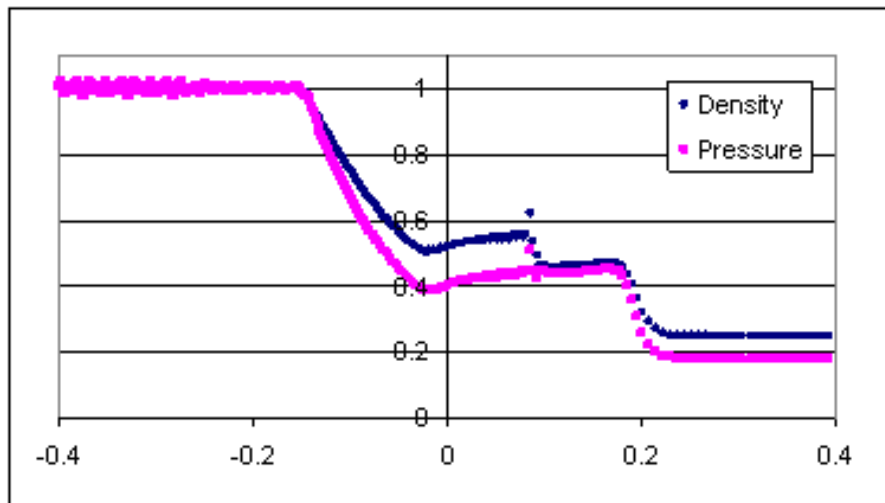
- Derivative of x

$$\frac{Dx_i}{Dt} = v_i$$

Simulation of Explosion (SPH)

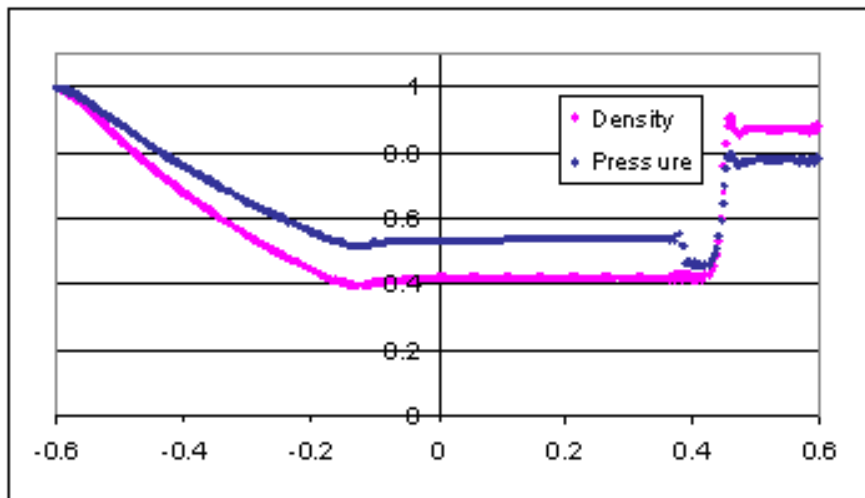


Simulation of Explosion (SPH)

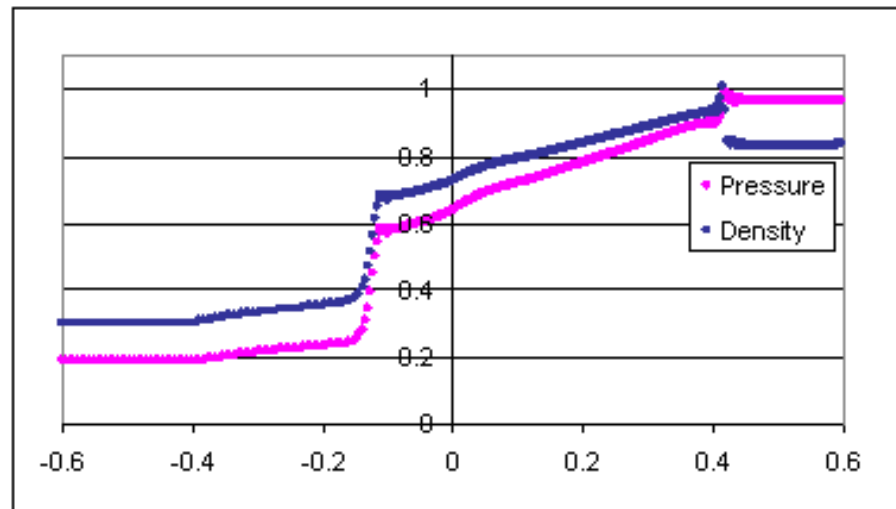


Simulation of Explosion (SPH)

Reflection of Shock from the end of tube



$t=0.55$



$t=1.20$

